GALOIS CORINGS APPLIED TO PARTIAL GALOIS THEORY

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ABSTRACT. Partial Galois extensions were recently introduced by Doku-chaev, Ferrero and Paques. We introduce partial Galois extensions for noncommutative rings, using the theory of Galois corings. We associate a Morita context to a partial action on a ring.

INTRODUCTION

Partial actions of groups originate from the theory of operator algebras, see for example [16]. Partial representations of groups on Hilbert spaces were introduced independently in [17] and [19]. Several applications are given in the literature, we refer to [14] for a more extensive bibliography. More recently, partial actions were studied from a purely algebraic point of view, in [12, 13, 15].

In [14], the authors consider partial actions on commutative rings, with the additional assumption that the associated ideals are generated by idempotents. Then they generalize Galois theory for commutative rings, as introduced in [10] for usual group actions, to partial actions.

Corings were introduced by Sweedler in 1975 in [23]. There has been a revived interest in corings since the beginning of the century, based on an observation made by Takeuchi that various types of modules, such as Hopf modules, relative Hopf modules, graded modules, entwined modules and Yetter-Drinfeld modules may be viewed as comodules over a coring. Brzeziński [1] noticed the importance of this observation: the language of corings can be applied successfully to give a unified and more elegant treatment to properties related to all these kinds of modules. An overview can be found in [4].

One of the nice applications is descent and Galois theory: Galois corings were introduced in [1], and studied in [6] and [24]. The corings approach provides a unified theory for various types of Galois theories, including the classical Chase-Harrison-Rosenberg theory [10], Hopf-Galois theory (see [11, 18, 20]), coalgebra Galois theory (see [3]) and weak Hopf-Galois theory (see the forthcoming [7]).

The aim of this note is to develop partial Galois theory starting from Galois corings. The strategy is basically the following: given a set of idempotents e_σ indexed by a finite group G in a ring A, we investigate when the direct

sum of the Ae_σ is a coring; it turns out that this is the case if a partial action of G on A is given. Then we investigate when this coring is a Galois coring, and apply the results in [6]. This procedure still works in the case where the ring A is not commutative. In the case where A is commutative, we recover some of the results in [14]. This is done in Section 2. In Section 3, we associate a Morita context to a partial action on a ring A, and show that the context is strict if A is a faithfully flat partial Galois extension of the invariants ring A^G.

1. Preliminary results

1.1. **Galois corings.** Let A be a ring. An A-coring $\mathcal{C}$ is a coalgebra in the category ${}_A\mathcal{M}_A$ of A-bimodules. Thus an A-coring is a triple $\mathcal{C} = (\mathcal{C}, \Delta_\mathcal{C}, \varepsilon_\mathcal{C})$, where $\mathcal{C}$ is an A-bimodule, and $\Delta_\mathcal{C} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and $\varepsilon_\mathcal{C} : \mathcal{C} \to A$ are A-bimodule maps such that

(1) $$(\Delta_\mathcal{C} \otimes_A \mathcal{C}) \circ \Delta_\mathcal{C} = (\mathcal{C} \otimes_A \Delta_\mathcal{C}) \circ \Delta_\mathcal{C},$$

and

(2) $$(\mathcal{C} \otimes_A \varepsilon_\mathcal{C}) \circ \Delta_\mathcal{C} = (\varepsilon_\mathcal{C} \otimes_A \mathcal{C}) \circ \Delta_\mathcal{C} = \mathcal{C}.$$

We use the Sweedler-Heyneman notation for the comultiplication:

$$\Delta_\mathcal{C}(c) = c_{(1)} \otimes_A c_{(2)}.$$

A right $\mathcal{C}$-comodule $M = (M, \rho)$ consists of a right A-module M together with a right A-linear map $\rho : M \to M \otimes_A \mathcal{C}$ such that:

(3) $$(\rho \otimes_A \mathcal{C}) \circ \rho = (M \otimes_A \Delta_\mathcal{C}) \circ \rho,$$

and

(4) $$(M \otimes_A \varepsilon_\mathcal{C}) \circ \rho = M.$$

We then say that $\mathcal{C}$ coacts from the right on M, and we denote

$$\rho(m) = m_{[0]} \otimes_A m_{[1]}.$$

A right A-linear map $f : M \to N$ between two right $\mathcal{C}$-comodules M and N is called right $\mathcal{C}$-colinear if $\rho(f(m)) = f(m_{[0]}) \otimes m_{[1]}$, for all $m \in M$. The category of right $\mathcal{C}$-comodules and $\mathcal{C}$-colinear maps is denoted by $\mathcal{M}^\mathcal{C}$.

$x \in \mathcal{C}$ is called grouplike if $\Delta_\mathcal{C}(x) = x \otimes x$ and $\varepsilon_\mathcal{C}(x) = 1$. Grouplike elements of $\mathcal{C}$ correspond bijectively to right $\mathcal{C}$-coactions on A: if A is grouplike, then we have the following right $\mathcal{C}$-coaction ρ on A: $\rho(a) = xa$.

Let $(\mathcal{C}, x)$ be a coring with a fixed grouplike element. For $M \in \mathcal{M}^\mathcal{C}$, we call

$$M^{\mathrm{co}\mathcal{C}} = \{m \in M \mid \rho(m) = m \otimes_A x\}$$

the submodule of coinvariants of M. Observe that

$$A^{\mathrm{co}\mathcal{C}} = \{b \in A \mid bx = xb\}$$

is a subring of A. Let $i : B \to A$ be a ring morphism. i factorizes through $A^{\mathrm{co}\mathcal{C}}$ if and only if

$$x \in G(\mathcal{C})^B = \{x \in G(\mathcal{C}) \mid xb = bx, \text{ for all } b \in B\}.$$

We then have a pair of adjoint functors (F, G), respectively between the categories $\mathcal{M}_B$ and $\mathcal{M}^{\mathcal{C}}$ and the categories $_B\mathcal{M}$ and $^{\mathcal{C}}\mathcal{M}$. For $N \in \mathcal{M}_B$ and $M \in \mathcal{M}^{\mathcal{C}}$,

$$F(N) = N \otimes_B A \text{ and } G(M) = M^{\mathrm{co}\mathcal{C}}.$$

The unit and counit of the adjunction are

$$\nu_N : N \to (N \otimes_B A)^{\mathrm{co}\mathcal{C}}, \ \nu_N(n) = n \otimes_B 1;$$

$$\zeta_M : M^{\mathrm{co}\mathcal{C}} \otimes_B A \to M, \ \zeta_M(m \otimes_B a) = ma.$$

Let $i : B \to A$ be a morphism of rings. The associated canonical coring is $\mathcal{D} = A \otimes_B A$, with comultiplication and counit given by the formulas

$$\Delta_{\mathcal{D}} : \mathcal{D} \to \mathcal{D} \otimes_A \mathcal{D} \cong A \otimes_B A \otimes_B A, \ \Delta_{\mathcal{D}}(a \otimes_B a') = a \otimes_B 1 \otimes_B a'$$

and

$$\varepsilon_{\mathcal{D}} : \mathcal{D} = A \otimes_B A \to A, \ \varepsilon_{\mathcal{D}}(a \otimes_B a') = aa'.$$

If $i : B \to A$ is pure as a morphism of left and right B-modules, then the categories $\mathcal{M}_B$ and $\mathcal{M}^{\mathcal{D}}$ are equivalent.

Let $(\mathcal{C}, x)$ be a coring with a fixed grouplike element, and $i : B \to A^{\mathrm{co}\mathcal{C}}$ a ring morphism. We then have a morphism of corings

$$\mathrm{can} : \mathcal{D} = A \otimes_B A \to \mathcal{C}, \ \mathrm{can}(a \otimes_B a') = axa'.$$

If F is fully faithful, then $B \cong A^{\mathrm{co}\mathcal{C}}$; if G is fully faithful, then can is an isomorphism. $(\mathcal{C}, x)$ is called a Galois coring if $\mathrm{can} : A \to_{A^{\mathrm{co}\mathcal{C}}} A \to \mathcal{C}$ is bijective. From [6], we recall the following results.

Theorem 1.1. *Let $(\mathcal{C}, x)$ be an A-coring with fixed grouplike element, and $B = A^{\mathrm{co}\mathcal{C}}$. Then the following statements are equivalent.*

(1) *$(\mathcal{C}, x)$ is Galois and A is faithfully flat as a left B-module;*
(2) *(F, G) is an equivalence and A is flat as a left B-module.*

Let $(\mathcal{C}, x)$ be a coring with a fixed grouplike element, and take $T = A^{\mathrm{co}\mathcal{C}}$. Then $^*\mathcal{C} = {}_A\mathrm{Hom}(\mathcal{C}, A)$ is a ring, with multiplication given by

$$(5) \qquad\qquad (f\#g)(c) = g(c_{(1)}f(c_{(2)})).$$

We have a morphism of rings $j : A \to {}^*\mathcal{C}$, given by

$$j(a)(c) = \varepsilon_{\mathcal{C}}(c)a.$$

This makes $^*\mathcal{C}$ into an A-bimodule, via the formula

$$(afb)(c) = f(ca)b.$$

Consider the left dual of the canonical map:

$$^*\mathrm{can} : {}^*\mathcal{C} \to {}^*\mathcal{D} \cong {}_T\mathrm{End}(A)^{\mathrm{op}}, \ {}^*\mathrm{can}(f)(a) = f(xa).$$

We then have the following result.

Proposition 1.2. *If $(\mathcal{C}, x)$ is Galois, then *can is an isomorphism. The converse property holds if $\mathcal{C}$ and A are finitely generated projective, respectively as a left A-module, and a left T-module.*

Let $Q = \{q \in {}^*\mathcal{C} \mid c_{(1)}q(c_{(2)}) = q(c)x,$ for all $c \in \mathcal{C}\}$. A straightforward computation shows that Q is a $({}^*\mathcal{C}, T)$-bimodule. Also A is a left $(T, {}^*\mathcal{C})$-bimodule; the right ${}^*\mathcal{C}$-action is induced by the right $\mathcal{C}$-coaction: $a \cdot f = f(xa)$. Now consider the maps

$$(6) \qquad\qquad \tau: \quad A \otimes_{{}^*\mathcal{C}} Q \to T, \ \tau(a \otimes_{{}^*\mathcal{C}} q) = q(xa);$$

$$(7) \qquad\qquad \mu: \quad Q \otimes_T A \to {}^*\mathcal{C}, \ \mu(q \otimes_T a) = q \# i(a).$$

With this notation, we have the following property (see [9]).

Proposition 1.3. $(T, {}^*\mathcal{C}, A, Q, \tau, \mu)$ *is a Morita context.*

We also have (see [6]):

Theorem 1.4. *Let $(\mathcal{C}, x)$ be a coring with fixed grouplike element, and assume that $\mathcal{C}$ is a left A-progenerator. We take a subring B of $T = A^{\mathrm{co}\mathcal{C}}$, and consider the map*

$$\mathrm{can}: \ \mathcal{D} = A \otimes_B A \to \mathcal{C}, \ \mathrm{can}(a \otimes_T a') = axa'$$

Then the following statements are equivalent:

- (1) • can *is an isomorphism;*
       • A *is faithfully flat as a left B-module.*
- (2) • *can *is an isomorphism;*
       • A *is a left B-progenerator.*
- (3) • $B = T$;
       • *the Morita context $(B, {}^*\mathcal{C}, A, Q, \tau, \mu)$ is strict.*
- (4) • $B = T$;
       • (F, G) *is an equivalence of categories.*

1.2. Partial group actions. Let G be a finite group, and $R \to S$ a commutative ring extension. From [13], we recall that a partial action α of G on S is a collection of ideals S_σ and isomorphisms of ideals $\alpha_\sigma: \ S_{\sigma^{-1}} \to S_\sigma$ such that

- (1) $S_1 = S$, and $\alpha_1 = S$, the identity on S;
- (2) $S_{(\sigma\tau)^{-1}} \supset \alpha_\tau^{-1}(S_\tau \cap S_{\sigma^{-1}})$;
- (3) $(\alpha_\sigma \circ \alpha_\tau)(x) = \alpha_{\sigma\tau}(x)$, for all $x \in \alpha_\tau^{-1}(S_\tau \cap S_{\sigma^{-1}})$.

In [14], the following particular situation is considered: every S_σ is of the form $S_\sigma = Se_\sigma$, where e_σ is an idempotent of S. In this case, we can show that

$$(8) \qquad\qquad \alpha_\sigma(\alpha_\tau(xe_{\tau^{-1}})e_{\sigma^{-1}}) = \alpha_{\sigma\tau}(xe_{\tau^{-1}\sigma^{-1}})e_\sigma,$$

for all $\sigma, \tau \in G$ and $x \in S$. We then have an associative ring with unit

$$A \star_\alpha G = \bigoplus_{\sigma \in G} Ae_\sigma u\sigma,$$

with multiplication

$$(9) \qquad (a_\sigma u_\sigma)(a_\tau u_\tau) = \alpha_\sigma(\alpha_{\sigma^{-1}}(a_\sigma)b_\tau)u_{\sigma\tau}.$$

2. PARTIAL GALOIS THEORY FOR NONCOMMUTATIVE RINGS

Let A be a (noncommutative) ring, and G a finite group. For every $\sigma \in G$, we assume that there is a central idempotent $e_\sigma \in A$, and a ring automorphism

$$\alpha_\sigma : \ Ae_{\sigma^{-1}} \to Ae_\sigma.$$

In particular, it follows that $\alpha_\sigma(e_{\sigma^{-1}}) = e_\sigma$. We can extend α_σ to A, by putting $\alpha_\sigma(a) = \alpha_\sigma(ae_\sigma)$, for all $a \in A$.

Then we consider the direct sum $\mathcal{C}$ of all the Ae_σ. Let v_σ be the element of $\mathcal{C}$ with e_σ in the Ae_σ-component, and 0 elsewhere. We then have

$$\mathcal{C} = \bigoplus_{\sigma \in G} Ae_\sigma v_\sigma = \bigoplus_{\sigma \in G} Av_\sigma.$$

Obviously $\mathcal{C}$ is a left A-module.

Lemma 2.1. $\mathcal{C}$ is an A-bimodule. The right A-action is given by the formula

$$(10) \qquad (a'v_\sigma)a = a'\alpha_\sigma(ae_{\sigma^{-1}})v_\sigma$$

Proof. Let us show that (10) is an associative action: for all $a, a' \in A$, we have

$$v_\sigma(aa') = \alpha_\sigma(aa'e_{\sigma^{-1}})v_\sigma = \alpha_\sigma(ae_{\sigma^{-1}}a'e_{\sigma^{-1}})v_\sigma$$
$$= \ \alpha_\sigma(ae_{\sigma^{-1}})\alpha_\sigma(a'e_{\sigma^{-1}})v_\sigma = \alpha_\sigma(ae_{\sigma^{-1}})v_\sigma a' = (v_\sigma a)a'.$$

$\square$

We now consider the left A-linear maps

$$\Delta_\mathcal{C} : \ \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}, \ \Delta_\mathcal{C}(av_\sigma) = \sum_{\tau \in G} av_\tau \otimes_A v_{\tau^{-1}\sigma};$$

$$\varepsilon_\mathcal{C} : \ \mathcal{C} \to A, \ \varepsilon_\mathcal{C}\left(\sum_{\sigma \in G} a_\sigma v_\sigma\right) = a_1.$$

Proposition 2.2. With notation as above, $(\mathcal{C}, \Delta_\mathcal{C}, \varepsilon_\mathcal{C})$ is an A-coring if and only if $e_1 = A$, $\alpha_1 = A$ and

$$(11) \qquad \alpha_\sigma(\alpha_\tau(ae_{\tau^{-1}})e_{\sigma^{-1}}) = \alpha_{\sigma\tau}(ae_{\tau^{-1}\sigma^{-1}})e_\sigma,$$

for all $a \in A$ and $\sigma, \tau \in G$.

Proof. We compute

$$
\begin{aligned}
\Delta_{\mathcal{C}}(v_\sigma a) &= \Delta_{\mathcal{C}}(\alpha_\sigma(ae_{\sigma^{-1}})v_\sigma \\
&= \sum_{\tau \in G} \alpha_\sigma(ae_{\sigma^{-1}})v_\tau \otimes_A v_{\tau^{-1}\sigma}; \\
\Delta_{\mathcal{C}}(v_\sigma)a &= \sum_{\tau \in G} v_\tau \otimes_A v_{\tau^{-1}\sigma}a \\
&= \sum_{\tau \in G} v_\tau \otimes_A \alpha_{\tau^{-1}\sigma}(ae_{\sigma^{-1}\tau})v_{\tau^{-1}\sigma} \\
&= \sum_{\tau \in G} \alpha_\tau(\alpha_{\tau^{-1}\sigma}(ae_{\sigma^{-1}\tau})e_{\tau^{-1}})v_\tau \otimes_A v_{\tau^{-1}\sigma}.
\end{aligned}
$$

Hence $\Delta_{\mathcal{C}}$ is right A-linear if and only if

$$
\alpha_\sigma(ae_{\sigma^{-1}})e_\tau = \alpha_\tau(\alpha_{\tau^{-1}\sigma}(ae_{\sigma^{-1}\tau})e_{\tau^{-1}}),
$$

for all $\sigma, \tau \in G$ and $a \in A$. Substituting $\lambda = \tau^{-1}\sigma$, we find that this is equivalent to (11).

Let us now investigate when $\varepsilon_{\mathcal{C}}$ is right A-linear. We have

$$
\varepsilon_{\mathcal{C}}(\sum_{\sigma \in G} a_\sigma v_\sigma)a = a_1 a)
$$

and

$$
\varepsilon_{\mathcal{C}}(\sum_{\sigma \in G} a_\sigma v_\sigma a) = \varepsilon_{\mathcal{C}}(\sum_{\sigma \in G} a_\sigma \alpha_\sigma(ae_{\sigma^{-1}})v_\sigma) = a_1 \alpha_1(ae_1)
$$

If $\varepsilon_{\mathcal{C}}$ is right A-linear, then we find that $\alpha_1(ae_1) = a$, for all $a \in A$. In particular,

$$
e_1 = \alpha_1(e_1 e_1) = \alpha_1(1e_1) = 1,
$$

and then it follows that $\alpha_1(a) = a$, for all $a \in A$. Conversely, if $e_1 = 1$ and $\alpha_1 = A$, then it follows that $\varepsilon_{\mathcal{C}}$ is right A-linear.

Now assume that (11) holds, and that $e_1 = 1$ and $\alpha_1 = A$. The coassociativity and counit property then follow in a straightforward way. $\square$

From now on, we will assume that $\mathcal{C} = \bigoplus_{\sigma \in G} Av_\sigma$ is an A-coring. The set of data $(e_\sigma, \alpha_\sigma)_{\sigma \in G}$ will be called an idempotent partial action of G on A. This is the case for the partial actions discussed in [14], that we recalled in Section 1.2, in view of (8).

Lemma 2.3. $x = \sum_{\sigma \in G} v_\sigma$ *is a grouplike element of* $\mathcal{C}$.

Proof. $\varepsilon_{\mathcal{C}}(x) = 1$, and

$$
\Delta_{\mathcal{C}}(x) = \sum_{\sigma, \tau \in G} v_\tau \otimes_A v_{\tau^{-1}\sigma} = \sum_{\rho, \tau \in G} v_\tau \otimes_A v_\rho = x \otimes_A x.
$$

$\square$

Consider the left A-linear maps

(12) $$u_\sigma : \ \mathcal{C} \to Ae_\sigma, \ u_\sigma(\sum_{\tau \in G} a_\tau v_\tau) = a_\sigma e_\sigma.$$

Then for all $c \in \mathcal{C}$, we have

$$c = \sum_{\sigma \in G} u_\sigma(c)v_\sigma,$$

hence $\{(u_\sigma, v_\sigma) \mid \sigma \in G\}$ is a dual basis of $\mathcal{C}$ as a left A-module.
Now let (M, ρ) be a right $\mathcal{C}$-comodule. We have a right A-linear map

$$\rho : \ M \to \bigoplus_{\sigma \in G} M \otimes_A Av_\sigma.$$

Consider the maps

$$\rho_\sigma = (M \otimes_A u_\sigma) \circ \rho : \ M \to Me_\sigma.$$

We then have

$$\rho(m) = m_{[0]} \otimes_A m_{[1]} = \sum_{\sigma \in G} \rho_\sigma(m) \otimes_A v_\sigma,$$

for all $m \in M$. From the fact ρ is right A-linear, it follows that

$$\rho(ma) = \sum_{\sigma \in G} \rho_\sigma(ma) \otimes_A v_\sigma = \rho(m)a = \sum_{\sigma \in G} \rho_\sigma(m) \otimes_A \alpha_\sigma(ae_{\sigma^{-1}})v_\sigma,$$

hence

(13) $$\rho_\sigma(ma) = \rho_\sigma(m)\alpha_\sigma(ae_{\sigma^{-1}}),$$

for all $m \in M$ and $\sigma \in G$. It follows from (13) that

$$\rho_\sigma(me_{\sigma^{-1}}) = \rho_\sigma(m)\alpha_\sigma(e_{\sigma^{-1}}) = \rho_\sigma(m)e_\sigma.$$

This means that $\rho_\sigma : \ M \to Me_\sigma$ factors through the projection $M \to Me_{\sigma^{-1}}$, so we obtain a map

$$\rho_\sigma : \ Me_{\sigma^{-1}} \to Me_\sigma.$$

Since $(M \otimes_A \varepsilon_\mathcal{C}) \circ \rho = M$, we have, for all $m \in M$:

$$m = \sum_{\sigma \in G} \rho_\sigma(m) \otimes_A \varepsilon_\mathcal{C}(v_\sigma) = \rho_1(m)e_1 = \rho_1(m).$$

Hence $\rho_1 : \ Me_1 = M \to Me_1 = M$ is the identity. From the coassociativity of ρ, we deduce that

$$\sum_{\sigma,\tau \in G} \rho_\tau(\rho_\sigma(m)) \otimes_A v_\tau \otimes_A v_\sigma = \sum_{\sigma,\rho \in G} \rho_\sigma(m) \otimes_A v_\mu \otimes_A v_{\mu^{-1}\sigma}$$

$$= \sum_{\kappa,\mu \in G} v_{\mu \circ \kappa} \otimes_A v_\mu \otimes_A v_\kappa = \sum_{\sigma,\tau \in G} v_{\tau \circ \sigma} \otimes_A v_\tau \otimes_A v_\sigma,$$

hence

(14) $$\rho_\tau(\rho_\sigma(m)) = \rho_{\tau \circ \sigma}(m),$$

for all $m \in M$ and $\sigma, \tau \in G$. In particular,

$$(15) \qquad \rho_\tau(\rho_\sigma(me_{\sigma^{-1}}e_{\tau^{-1}})) = \rho_{\tau \circ \sigma}(me_{\sigma^{-1}\tau^{-1}})e_\tau.$$

It follows from (15) that $\rho_{\tau^{-1}} :\ Me_\tau \to Me_{\tau^{-1}}$ is the inverse of $\rho_\tau :\ Me_{\tau^{-1}} \to Me_\tau$.

Definition 2.4. Let $(e_\sigma, \alpha_\sigma)_{\sigma \in G}$ be an idempotent partial action of G on A, and M a right A-module. A partial Galois descent datum consists of a set of maps

$$\rho_\sigma :\ M \to Me_\tau$$

such that $\rho_1 = M$, the identity on M, the restriction of ρ_σ to $Me_{\tau^{-1}}$ is an isomorphism, and (13) and (15) hold for all $m \in M$, $a \in A$ and $\sigma, \tau \in G$.

Proposition 2.5. *Let $(e_\sigma, \alpha_\sigma)_{\sigma \in G}$ be an idempotent partial action of G on A, and $\mathcal{C}$ the corresponding A-coring. Then right $\mathcal{C}$-coactions on M correspond bijectively to partial Galois descent data.*

Proof. We have already explained above how a right $\mathcal{C}$-coaction ρ on M can be transformed into a partial Galois descent datum. Conversely, let $(\rho_\sigma)_{\sigma \in G}$ be a partial Galois descent datum, and define $\rho :\ M \to M \otimes_A \mathcal{C}$ by

$$\rho(m) = \sum_{\sigma \in G} \rho_\sigma(m) \otimes_A v_\sigma.$$

Straightforward computations show that ρ is a coaction, and that the two constructions are inverse to each other. $\qquad\qquad\square$

Let M be a right $\mathcal{C}$-comodule. Then $m \in M^{\mathrm{co}\mathcal{C}}$ if and only if

$$\rho(m) = \sum_{\sigma \in G} \rho_\sigma(m) \otimes_A v_\sigma = \sum_{\sigma \in G} m \otimes_A v_\sigma = \sum_{\sigma \in G} me_\sigma \otimes_A v_\sigma$$

if and only if

$$\rho_\sigma(m) = \rho_\sigma(me_{\sigma^{-1}}) = me_\sigma,$$

for all $\sigma \in G$. We define

$$M^G = \{m \in M \mid \rho_\sigma(me_{\sigma^{-1}}) = me_{\sigma^{-1}},\ \text{for all } \sigma \in G\} = M^{\mathcal{C}}.$$

The grouplike element $x = \sum_{\sigma \in G} v_\sigma$ makes A into a right $\mathcal{C}$-comodule:

$$\rho(a) = 1 \otimes_A xa = \sum_{\sigma \in G} \alpha_\sigma(ae_{\sigma^{-1}}) \otimes_A v_\sigma,$$

and we have

$$T = A^G = \{a \in A \mid \alpha_\sigma(ae_{\sigma^{-1}}) = ae_\sigma,\ \text{for all } \sigma \in G\}.$$

Let $i :\ B \to T$ be a ring morphism. We have seen in Section 1.1 that we have a pair of adjoint functors (F, G):

$$F :\ \mathcal{M}_B \to \mathcal{M}^{\mathcal{C}},\ F(N) = N \otimes_B A;$$

$$G :\ \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_B,\ G(N) = N^G.$$

$F(N) = N \otimes_B A$ is a right $\mathcal{C}$-comodule in the following way:

$$\rho_\sigma(n \otimes_A a) = n \otimes \alpha_\sigma(a).$$

The canonical map is the following:

$$\text{can}: \ A \otimes_B A \to \bigoplus_{\sigma \in G} A_e v_\sigma, \ \text{can}(a \otimes b) = \sum_{\sigma \in G} a \alpha_\sigma(be_{\sigma^{-1}}) v_\sigma.$$

$\bigoplus_{\sigma \in G} A_e v_\sigma$ is a Galois coring if $\text{can}: A \otimes_{A^G} A \to \bigoplus_{\sigma \in G} A_e v_\sigma$ is an isomorphism. We will then say that A is a partial G-Galois extension of A^G. From Theorem 1.1, we immediately obtain the following result.

Theorem 2.6. *Let $(e_\sigma, \alpha_\sigma)_{\sigma \in G}$ be an idempotent partial action of G on A, and $T = A^G$. Then the following assertions are equivalent.*
  (1) *A is a partial G-Galois extension of T and T is faithfully flat as a left T-module;*
  (2) *(F, G) is a category equivalence and A is flat as a left T-module.*

3. Partial actions and Morita theory

Let us now compute the multiplication on

$$^*\mathcal{C} = {}_A\text{Hom}(\mathcal{C}, A) = \bigoplus_{\sigma \in G} {}_A\text{Hom}(A e_\sigma, A).$$

We will use the maps u_σ defined in (12). Also recall that $^*\mathcal{C}$ is an A-bimodule, with left and right A-action

$$(afb)(c) = f(ca)b.$$

Take $f \in {}_A\text{Hom}(\mathcal{C}, A)$. For all $c \in \mathcal{C}$, we have

$$f(c) = \sum_{\sigma \in \mathcal{C}} u_\sigma(c) f(v_\sigma).$$

Now $f(v_\sigma) = f(e_\sigma v_\sigma) = e_\sigma f(v_\sigma) \in A e_\sigma$, so we can conclude that

$$^*\mathcal{C} = \bigoplus_{\sigma \in \mathcal{C}} u_\sigma A e_\sigma.$$

For $b \in A$, we compute

$$(bu_\tau)(\sum_{\sigma \in G} a_\sigma v_\sigma) = u_\tau(\sum_{\sigma \in G} a_\sigma v_\sigma b) = u_\tau(\sum_{\sigma \in G} a_\sigma \alpha_\sigma(be_{\sigma^{-1}})v_\sigma) = a_\tau \alpha_\tau(be_{\tau^{-1}}),$$

and we conclude that

$$(16) \qquad\qquad bu_\tau = u_\tau \alpha_\tau(be_{\tau^{-1}}).$$

We next compute, using (11):

$$(u_\rho \# u_\nu)(\sum_{\sigma \in G} a_\sigma v_\sigma) = u_\nu(\sum_{\sigma,\tau} a_\sigma v_\tau u_\rho(v_{\tau^{-1}\sigma})) = u_\nu(\sum_{\sigma,\tau} a_\sigma v_\tau \delta_{\tau\rho,\sigma})$$

$$= \ u_\nu(\sum_{\tau} a_{\tau\rho} v_\tau) = a_{\nu\rho} = u_{\nu\rho}(\sum_{\sigma \in G} a_\sigma v_\sigma),$$

and we conclude that

$$(17) \qquad\qquad u_\sigma \# u_\tau = u_{\sigma\tau}.$$

We can summarize this as follows:

Proposition 3.1. *Let* $\mathcal{C} = \bigoplus_{\sigma \in G} A v_\sigma$. *The left dual ring is*

$$^*\mathcal{C} = \bigoplus_{\sigma \in G} u_\sigma A e_\sigma,$$

with multiplication rule

$$(18) \qquad\qquad u_\tau b_\tau \# u_\sigma a_\sigma = u_{\sigma\tau} \alpha_\sigma(b_\tau e_{\sigma^{-1}}) a_\sigma.$$

If A is commutative, then $^*\mathcal{C}$ is isomorphic to $(A \star_\alpha G)^{\mathrm{op}}$, as introduced in [14], see (9). Indeed, for $a \in A e_\sigma$ and $b \in A e_\tau$, we compute that

$$\alpha_\sigma(\alpha_{\sigma^{-1}}(a_\sigma) b_\tau) = \alpha_\sigma(\alpha_{\sigma^{-1}}(a_\sigma) b_\tau e_{\sigma^{-1}})$$
$$= a_\sigma \alpha_\sigma(b_\tau e_{\sigma^{-1}}) \alpha_\sigma(b_\tau e_{\sigma^{-1}}) a_\sigma$$

Recall that a ring morphism $A \to R$ is called *Frobenius* if there exists an A-bimodule map $\overline{\nu} : R \to A$ and $e = e^1 \otimes_A e^2 \in R \otimes_A R$ (summation implicitly understood) such that

$$(19) \qquad\qquad r e^1 \otimes_A e^2 = e^1 \otimes_A e^2 r$$

for all $r \in R$, and

$$(20) \qquad\qquad \overline{\nu}(e^1) e^2 = e^1 \overline{\nu}(e^2) = 1.$$

This is equivalent to the restrictions of scalars $\mathcal{M}_R \to \mathcal{M}_A$ being a Frobenius functor, which means that its left and right adjoints are isomorphic (see [8, Sec. 3.1 and 3.2]). $(e, \overline{\nu})$ is then called a Frobenius system.

Proposition 3.2. *Suppose that we have an idempotent partial action of G on A. Then the ring morphism $A \to {}^*\mathcal{C}$ is Frobenius.*

Proof. The Frobenius system is $(e = \sum_{\sigma \in G} u_{\sigma^{-1}} \otimes_A u_\sigma, \overline{\nu})$, with

$$\overline{\nu}\Big(\sum_{\sigma \in G} u_\sigma a_\sigma\Big) = a_1.$$

We compute that, for all $a \in A$,

$$a \sum_{\sigma \in G} u_{\sigma^{-1}} \otimes_A u_\sigma = \sum_{\sigma \in G} u_{\sigma^{-1}} \alpha_{\sigma^{-1}}(a e_\sigma) \otimes_A u_\sigma$$
$$= \sum_{\sigma \in G} u_{\sigma^{-1}} \otimes_A u_\sigma \alpha_\sigma(\alpha_{\sigma^{-1}}(a e_\sigma)) = \sum_{\sigma \in G} u_{\sigma^{-1}} \otimes_A u_\sigma a e_\sigma$$
$$= \sum_{\sigma \in G} u_{\sigma^{-1}} \otimes_A u_\sigma a.$$

The rest is obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $i : B \to T = A^{\mathrm{co}\mathcal{C}}$ be a ring morphism. We have the canonical morphism

$$\mathrm{can} : \mathcal{D} = A \otimes_B A \to \mathcal{C} = \bigoplus_{\sigma \in G} A v_\sigma,$$

given by

$$\mathrm{can}(a \otimes b) = \sum_{\sigma \in G} a v_\sigma b = \sum_{\sigma \in G} a \alpha_\sigma(b e_{\sigma^{-1}}) v_\sigma$$

We can also compute that

$$^*\mathrm{can} : \, {}^*\mathcal{C} = \bigoplus_{\sigma \in G} u_\sigma A \to {}^*\mathcal{D} \cong {}_B\mathrm{End}(A)^{\mathrm{op}}$$

is given by

$$^*\mathrm{can}(u_\tau b_\tau)(a) = \alpha_\tau(a e_{\tau^{-1}}) b_\tau.$$

Let us now compute the module $Q \subset {}^*\mathcal{C}$ introduced in Section 1.1 Recall that $q \in Q$ if and only if

$$(21) \qquad c_{(1)} q(c_{(2)}) = q(c)x,$$

for all $c \in \mathcal{C}$.

Proposition 3.3. $Q = \{\sum_{\sigma \in G} u_\sigma \alpha_\sigma(a e_{\sigma^{-1}}) \mid a \in A\}$.

Proof. Take $q = \sum_{\sigma \in G} u_\sigma a_\sigma \in Q$, with $a_\sigma \in A e_\sigma$, and put $c = v_\tau$ in (21). Recall that $\Delta_\mathcal{C}(v_\sigma) = \sum_{\rho \in G} v_\rho \otimes_A v_{\rho^{-1}\tau}$. Then we calculate that

$$c_{(1)} q(c_{(2)}) = \sum_{\rho,\sigma \in G} v_\rho \delta_{\sigma,\rho^{-1}\tau} a_\sigma = \sum_{\rho \in G} v_\rho a_{\rho^{-1}\tau} = \sum_{\rho \in G} \alpha_\rho(a_{\rho^{-1}\tau} e_{\rho^{-1}}) v_\rho,$$

$$q(c) = \Big(\sum_{\sigma \in G} u_\sigma a_\sigma\Big)(v_\tau) = a_\tau$$

and

$$q(c)x = \sum_{\rho \in G} a_\tau v_\rho$$

Hence it follows that

$$(22) \qquad a_\tau e_\rho = \alpha_\rho(a_{\rho^{-1}\tau} e_{\rho^{-1}}),$$

for all $\tau, \rho \in G$. Taking $\tau = \rho$, we find that

$$(23) \qquad a_\tau = a_\tau e_\tau = \alpha_\tau(a_1 e_{\tau^{-1}}),$$

and we find that

$$(24) \qquad q = \sum_{\sigma \in G} u_\sigma a_\sigma = \sum_{\sigma \in G} u_\sigma \alpha_\sigma(a_1 e_{\sigma^{-1}})$$

is of the desired form. Conversely, take q of the form (24). Then (23) holds. Using (11), we compute

$$\alpha_\rho(a_{\rho^{-1}\tau} e_{\rho^{-1}}) = \alpha_\rho(\alpha_{\rho^{-1}\tau}(a_1 e_{\tau^{-1}\rho}) e_{\rho^{-1}}) = \alpha_\tau(a_1 e_{\tau^{-1}}) e_\rho = a_\tau e_\rho,$$

and (22) follows, which means that (21) holds for $c = v_\tau$. Using the left A-linearity of q and $\Delta_\mathcal{C}$, it follows that (21) holds for arbitrary $c \in \mathcal{C}$. $\square$

It follows from Proposition 3.3 that we have an isomorphism of abelian groups

$$A \to Q, \ a \mapsto \sum_{\sigma \in G} u_\sigma \alpha_\sigma(ae_{\sigma^{-1}}).$$

This can also be seen using Proposition 3.2 and [9, Theorem 2.7]. The $(^*\mathcal{C}, T)$-bimodule structure on Q (see Proposition 1.2) can be transported to A. The right T-action on A is then given by right multiplication, and the left $^*\mathcal{C}$-action is the following:

$$(u_\tau a_\tau) \cdot a = \alpha_{\tau^{-1}}(a_\tau ae_\tau).$$

Recall also from Proposition 1.2 that $A \in {}_T\mathcal{M}_{*\mathcal{C}}$. The left T-action is given by left multiplication. The right $^*\mathcal{C}$-action is the following:

$$a \cdot (u_\tau a_\tau) = (au_\tau a_\tau)(x)$$
$$= \sum_{\sigma \in G} u_\tau \alpha_\tau(ae_{\tau^{-1}})a_\tau)(v_\sigma) = \alpha_\tau(ae_{\tau^{-1}})b_\tau.$$

We have seen in Proposition 1.2 that we have a Morita context $(T, {}^*\mathcal{C}, A, Q, \tau, \mu)$. Using the isomorphism between A and Q, we find a Morita context $(T, {}^*\mathcal{C}, A, A, \tau, \mu)$. Let us compute the connecting maps $\tau : \ A \otimes_{*\mathcal{C}} A \to T$ and $\mu : \ A \otimes_T A \to {}^*\mathcal{C}$, using (6-7).

$$\tau(b \otimes a) = (\sum_{\sigma \in G} u_\sigma \alpha_\sigma(ae_{\sigma^{-1}}))(\sum_{\tau \in G} (v_\tau b))$$
$$= \sum_{\sigma, \tau \in G} u_\sigma(\alpha_\tau(be_{\tau^{-1}})v_\tau)\alpha_\sigma(ae_{\sigma^{-1}})$$
$$= \sum_{\sigma \in G} \alpha_\sigma(be_{\sigma^{-1}})\alpha_\sigma(ae_{\sigma^{-1}})$$
$$= \sum_{\sigma \in G} \alpha_\sigma(bae_{\sigma^{-1}});$$
$$\mu(a \otimes b) = \sum_{\sigma \in G} u_\sigma \alpha_\sigma(ae_{\sigma^{-1}})b.$$

We summarize our results as follows.

Proposition 3.4. *We have a Morita context* $(T, {}^*\mathcal{C}, A, Q, \tau, \mu)$. *The connecting maps are given by the formulas*

$$(25) \qquad\qquad \tau(b \otimes a) \ = \ \sum_{\sigma \in G} \alpha_\sigma(bae_{\sigma^{-1}});$$

$$(26) \qquad\qquad \mu(a \otimes b) \ = \ \sum_{\sigma \in G} u_\sigma \alpha_\sigma(ae_{\sigma^{-1}})b.$$

Proposition 3.5. *The map* τ *in the Morita context from Proposition 3.4 is surjective if and only if there exists* $a \in A$ *such that*

$$\sum_{\sigma \in G} \alpha_\sigma(ae_{\sigma^{-1}}) = 1.$$

Proof. According to [9, Theorem 3.3], τ is surjective if and only if there exists $q \in Q$ such that $q(x) = 1$. Let $a \in A$ correspond to $q \in Q$. Then we compute that

$$q(x) = (\sum_{\sigma \in G} u_\sigma \alpha_\sigma(ae_{\sigma^{-1}}))(\sum_{\tau \in G} v_\tau) = \sum_{\sigma \in G} \alpha_\sigma(ae_{\sigma^{-1}}),$$

and the result follows. □

From Theorem 1.4, we obtain:

Theorem 3.6. *Let G be a finite group, and $(e_\sigma, \alpha_\sigma)_{\sigma \in G}$ an idempotent partial action of G on A. Let $i: B \to T = A^{\mathrm{co}\mathcal{C}}$ a ring morphism, and consider* can : $A \otimes_B A \to \mathcal{C}$. *Then the following assertions are equivalent.*

- (1)
  - can *is an isomorphism;*
  - *A is faithfully flat as a left B-module.*
- (2)
  - *can *is an isomorphism;*
  - *A is a left B-progenerator.*
- (3)
  - *$B = T$;*
  - *the Morita context $(B, {}^*\mathcal{C}, A, A, \tau, \mu)$ is strict.*
- (4)
  - *$B = T$;*
  - *(F, G) is an equivalence of categories.*

If we take A and B commutative, then Theorem 3.6 implies part of [14, Theorem 3.1], namely the equivalence of the conditions (i), (ii) and (iii).

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